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On Sboxes sharing the same DDT

Christina Boura, Anne Canteaut, Jérémy Jean, Valentin Suder

University of Versailles, France

Inria Paris, France

Anssi, France

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Problem

Find all n -bit Sboxes having a given difference distribution table.

α/β	0	1	2	3	4	5	6	7
0	8
1	.	2	.	2	.	2	.	2
2	.	.	2	2	.	.	2	2
3	.	2	2	.	.	2	2	.
4	2	2	2	2
5	.	2	.	2	2	.	2	.
6	.	.	2	2	2	2	.	.
7	.	2	2	.	2	.	.	2

where $\delta_F(\alpha, \beta) = \#\{x \in \mathbb{F}_2^n : F(x + \alpha) + F(x) = \beta\}$

Some trivial properties of the DDT

Differential uniformity of F :

$$\delta(F) = \max_{\alpha \neq 0, \beta} \delta_F(\alpha, \beta) .$$

$\delta(F) \geq 2$ with equality for **APN functions**.

- All entries in the DDT are even.
- The entries in a row sum to 2^n .

Related open problems

- Characterization of valid DDTs.
- Characterization of the functions sharing the same DDT.
- **The big APN problem** [Dillon 09]: Does there exist an APN permutation of n variables with n even, $n \geq 8$?
- **The crooked conjecture** [Bending, Fon-der-Flaass 98]:
 F is an APN permutation of degree 2 if and only if the support of every row in the DDT is the complement of a hyperplane.

Indicator of the DDT [Carlet, Charpin, Zinoviev 98]

Definition:

For any n -bit Sbox F , γ_F is the Boolean function of $2n$ variables defined by

$$\gamma_F(\alpha, \beta) = 0 \text{ if and only if } \delta_F(\alpha, \beta) = 0 \text{ or } \alpha = 0 .$$

α/β	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	*	0	*	0	*	0	*
2	0	0	*	*	0	0	*	*
3	0	*	*	0	0	*	*	0
4	0	0	0	0	*	*	*	*
5	0	*	0	*	*	0	*	0
6	0	0	*	*	*	*	0	0
7	0	*	*	0	*	0	0	*

Two notions of differential equivalence

- DDT-equivalence:

$$F \sim_{\text{DDT}} G \quad \Leftrightarrow \quad \text{DDT}_F = \text{DDT}_G$$

- γ -equivalence (aka differential equivalence [Gorodilova 16]):

$$F \sim_{\gamma} G \quad \Leftrightarrow \quad \gamma_F = \gamma_G$$

DDT-equivalence \Rightarrow γ -equivalence

The two notions are different

For $n = 4$

$$F = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 14],$$

$$G = [0, 1, 3, 2, 5, 4, 7, 6, 8, 9, 10, 11, 12, 13, 14, 15].$$

[illegible]

The two notions are different

For $n = 4$

$$F = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 14],$$

$$G = [0, 1, 3, 2, 5, 4, 7, 6, 8, 9, 10, 11, 12, 13, 14, 15].$$

$$\text{DDT}_G = \begin{bmatrix} 16 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 16 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & 4 \end{bmatrix}$$

γ -equivalence and differential uniformity

The following 4-bit Sboxes are γ -equivalent:

$F_1 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ with $\delta(F_1) = 14$,

$F_2 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1]$ with $\delta(F_2) = 12$,

$F_3 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1]$ with $\delta(F_3) = 10$,

$F_4 = [1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1]$ with $\delta(F_4) = 8$.

The two notions coincide in some cases

α/β	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	*	0	*	0	*	0	*
2	0	0	*	*	0	0	*	*
3	0	*	*	0	0	*	*	0
4	0	0	0	0	*	*	*	*
5	0	*	0	*	*	0	*	0
6	0	0	*	*	*	*	0	0
7	0	*	*	0	*	0	0	*

$F \sim_\gamma G$ implies $F \sim_{\text{DDT}} G$

- when F is **APN**.
- when F and G are **quadratic**.

Sboxes sharing the same DDT

Trivially equivalent Sboxes

Proposition.

The DDT-equivalence class of an n -bit Sbox contains all functions of the form

$$x \longmapsto F(x + \textcolor{red}{c}) + \textcolor{red}{d}, \text{ for } \textcolor{red}{c}, \textcolor{red}{d} \in \mathbb{F}_2^n.$$

We say that the DDT-equivalence class of F is **trivial** if it contains trivially equivalent Sboxes only.

Problems.

- Characterize the Sboxes having a trivial DDT-equivalence class.
- Determine the properties of the Sboxes within a non-trivial DDT-equivalence class.

An equivalent formulation

F and G share the same DDT iff they share the same squared LAT.

\Rightarrow Sboxes within the same DDT-equivalence class correspond to LAT with different sign sequences:

$$\mathcal{W}_G(\lambda, \mu) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\lambda \cdot G(x) + \mu \cdot x} = (-1)^{s(\lambda, \mu)} \mathcal{W}_F(\lambda, \mu).$$

F and G are trivially DDT-equivalent Sboxes if and only if

$$s(\lambda, \mu) = d \cdot \lambda + c \cdot \mu.$$

Algebraic degree of DDT-equivalent Sboxes

Conjecture [Gorodilova 16]. If F is a quadratic APN Sbox, then any G in the DDT-class of F satisfies

$$\deg(F + G) \leq 1 .$$

In general: For any even n , all n -bit Sboxes defined by

$$S(x) = (f(x), c_1, \dots, c_{n-1})$$

where f is a bent function and (c_1, \dots, c_{n-1}) is a constant, have the same DDT.

All rows are equal to

$$[2^{n-1}, 2^{n-1}, 0, 0, \dots, 0]$$

\Rightarrow There exist Sboxes of any degree between 2 and $n/2$ in this DDT-equivalence class.

Extended affine equivalence

Definition.

Two n -bit Sboxes F and G are extended-affine (EA) equivalent if there exist affine functions A_0 , A_1 , A_2 , where A_1 and A_2 are bijective such that

$$G = A_1 \circ F \circ A_2 + A_0 .$$

Proposition (adapted from [Gorodilova 16])

If F and G are EA-equivalent, then their DDT-equivalence (resp. γ -equivalence) classes have the same size.

$$\mathcal{C}_{\text{DDT}}(G) = \{A_1 \circ F' \circ A_2 + A_0, \text{ with } F' \in \mathcal{C}_{\text{DDT}}(F)\}$$

CCZ equivalence [Carlet, Charpin, Zinoviev 98]

Definition.

Two n -bit Sboxes F and G are said **CCZ equivalent** if

$\{(y, G(y)), y \in \mathbb{F}_2^n\}$ is the image of $\{(x, F(x)), x \in \mathbb{F}_2^n\}$

by a **linear permutation** \mathcal{L} of $\mathbb{F}_2^n \times \mathbb{F}_2^n$.

In particular, if $\mathcal{L} : (x, y) \mapsto (L_1(x, y), L_2(x, y))$, then

$x \mapsto L_1(x, F(x))$ is a **permutation**.

Open problem [Gorodilova 16]

Does an analogue of the result for **EA** equivalence hold for **CCZ** equivalence?

CCZ equivalence

Theorem.

If F and G are CCZ-equivalent then

- their DDT-equivalence (resp. γ -equivalence) classes have the same size.
- The DDT-class of G is obtained by applying the same linear permutation \mathcal{L} to all functions in the DDT-class of F .

Algorithm for determining all Sboxes having a prescribed γ

Input : The indicator of a DDT,

Output : All functions having this indicator

Idea: Recursive Tree-traversal algorithm

- Tree of depth 2^n : each node at level i corresponds to one possible value for $F(i)$.
- From the constraints of the DDT and the values $F(0), \dots, F(i-1)$:
 - find all possible values for $F(i)$
 - for each of them, move on to the next step $F(i+1)$, and backtrack if necessary

Pruning trick: We fix

$$F(0) \text{ and } F(1)$$

Experimental results

Permutations with optimal differential uniformity

APN permutations.

The DDT-equivalence classes of all known APN permutations for $n \leq 9$ are trivial.

Optimal permutations for $n = 4$.

The DDT-equivalence classes and the γ -equivalence classes of all permutations F with $\delta(F) = 4$ and optimal linearity listed in [Leander, Poschmann 07] are trivial.

APN non-bijective functions

The DDT-equivalence classes of all known APN functions for $n \leq 8$ are trivial, except:

- when $n \equiv 0 \pmod{4}$: the Gold APN functions with exponents $2^k + 1$ with $k = n/2 \pm 1$ [Gorodilova 16]
- for $n = 6$: Class 13 in [Brinckmann, Leander 08].

We checked that, for $n = 6$, all APN functions of degree ≤ 3 are trivial except Class 13.

Do all permutations have a trivial DDT class?

The following permutations share the same DDT.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$F(x)$	0	1	2	3	4	5	6	7	8	9	10	11	13	12	15	14	16
$G(x)$	0	1	2	3	4	5	6	7	8	9	10	11	13	12	15	14	16

x	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$F(x)$	17	19	18	20	21	23	22	25	24	26	27	28	29	31	30
$G(x)$	17	19	18	21	20	22	23	24	25	27	26	28	29	31	30

Some conclusions and a conjecture

- All Sboxes we found with a non-trivial DDT-equivalence class have non-distinct rows in their DTT.
- All rows in the DDT of an APN permutation are distinct.

Conjecture.

The DDT-equivalence class of any APN permutation is trivial.

Open problem.

Find a family of Sboxes for which it can be proved that the DDT-equivalence class is trivial.

Toy example for $n = 3$

$$\mathcal{R}_i := \{j \mid \mathbf{DDT}(i, j) \neq 0\}.$$

	0	1	2	3	4	5	6	7
0	8
1	.	2	.	2	.	2	.	2
2	.	.	2	2	.	.	2	2
3	.	2	2	.	.	2	2	.
4	2	2	2	2
5	.	2	.	2	2	.	2	.
6	.	.	2	2	2	2	.	.
7	.	2	2	.	2	.	.	2

- Set $F(0) = 0$
- $F(0) + F(1) \in \mathcal{R}_1 = \{1, 3, 5, 7\} \Rightarrow$ Set $F(1) = 1$.
- $F(0) + F(2) \in \mathcal{R}_2 = \{2, 3, 6, 7\}$ and $F(1) + F(2) \in \mathcal{R}_3 = \{1, 2, 5, 6\} \Rightarrow F(2) \in \{3, 7\}$

Toy example for $n = 3$

